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Polynomials Associated with Equilibria of Affine Toda-Sutherland Systems

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Abstract

An affine Toda-Sutherland system is a *quasi-exactly solvable* multi-particle dynamics based on an affine simple root system. It is a 'cross' between two well-known integrable multi-particle dynamics, an affine Toda molecule and a Sutherland system. Polynomials describing the equilibrium positions of affine Toda-Sutherland systems are determined for all affine simple root systems.

1 Introduction

Given a multi-particle dynamical system, to find and describe its equilibrium position has practical as well as theoretical significance. As is well-known, near the equilibrium the system is reduced to a collection of harmonic oscillators and that their spectra give the exact order $\hbar$ part of the full quantum spectra [1]. Naively, one could describe the equilibrium position by zeros of a certain polynomial. In this way one obtains the celebrated classical orthogonal polynomials for *exactly solvable* multi-particle dynamics. For the Calogero systems [2] based on the A and B (C, BC and D) root systems, the equilibrium positions correspond to the

zeros of the Hermite and Laguerre polynomials [3, 4, 5, 6]. For the Sutherland systems [7] based on the A and B (C, BC and D) root systems, the equilibrium positions correspond to the zeros of the Chebyshev and Jacobi polynomials [6]. Polynomials describing the equilibria of the Calogero and Sutherland systems based on the exceptional root systems are also determined [8]. In all these cases the frequencies of small oscillations at the equilibrium are "*quantised*" [6, 9]. For another family of multi-particle dynamics based on root systems, the Ruijsenaars-Schneider systems [10], which are *deformation* of the Calogero and Sutherland systems, the corresponding polynomials are determined [11, 12]. They turn out to be *deformation* of the Hermite, Laguerre and Jacobi polynomials which inherit the orthogonality [12]. The frequencies of small oscillations at the equilibrium are also "*quantised*" [11]. Another interesting feature is that the equations determining the equilibrium look like *Bethe ansatz* equations.

One is naturally led to a similar investigation for partially solvable or *quasi-exactly solvable* [13] multi-particle dynamics. From a not-so-long list of known quasi-exactly solvable multi-particle dynamical systems [14], we pick up the so-called affine Toda-Sutherland systems [15] and determine polynomials describing the equilibrium positions. These polynomials, as well as all the polynomials mentioned above, are characterised as having *integer* coefficients only.

2 affine Toda-Sutherland systems

The affine Toda-Sutherland systems are quasi-exactly solvable [13] multi-particle dynamics based on any crystallographic root system. Roughly speaking, they are obtained by 'crossing' two well-known integrable dynamics, the affine-Toda molecule and the Sutherland system. Given a set of *affine simple roots* $\Pi_0 = \{\alpha_0, \alpha_1, \ldots, \alpha_r\}$, $\alpha_j \in \mathbb{R}^r$, let us introduce a *prepotential* W [16]

$$W(q) = g \sum_{j=0}^{r} n_j \log|\sin(\alpha_j \cdot q)|, \quad q = {}^t(q_1, \ldots, q_r) \in \mathbb{R}^r, \tag{1}$$

in which g is a positive coupling constant and $\{n_j\}$ are the Dynkin-Kac labels for Π_0. That is, they are the integer coefficients of the affine simple root α_0; $-\alpha_0 = \sum_{j=1}^{r} n_j \alpha_j$, $n_0 \equiv 1$. For simply-laced and un-twisted non-simply laced affine root systems α_0 is the lowest long root, whereas for twisted non-simply laced affine root systems, α_0 is the lowest short root.

In either case $h \stackrel{\text{def}}{=} \sum_{j=0}^{r} n_j$ is the *Coxeter number*. This leads to the classical Hamiltonian

$$H_C = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\partial W(q)}{\partial q_j}\right)^2. \tag{2}$$

It is shown [15] that the equilibrium position $\bar{q}$ is given by a *universal* formula in terms of the dual Weyl vector $\varrho^\vee$:

$$\frac{\partial W(\bar{q})}{\partial q_j} = 0 \quad \Leftrightarrow \quad \bar{q} = \frac{\pi}{h}\varrho^\vee, \qquad \varrho^\vee \stackrel{\text{def}}{=} \sum_{j=1}^{r} \lambda_j^\vee. \tag{3}$$

The dual fundamental weight $\lambda_j^\vee$ is defined in terms of the fundamental weight λ_j by $\lambda_j^\vee \stackrel{\text{def}}{=} (2/\alpha_j^2)\lambda_j$, which satisfies $\alpha_j \cdot \lambda_k^\vee = \delta_{j\,k}$. At the equilibrium, the classical multi-particle dynamical system (2) is reduced to a set of harmonic oscillators. The frequencies (not frequencies squared) of small oscillations at the equilibrium of the affine Toda-Sutherland model are given up to the coupling constant g by [15]

$$\frac{1}{\sin^2\frac{\pi}{h}}\left\{m_1^2, m_2^2, \ldots, m_r^2\right\},$$

in which m_j^2 are the so-called affine Toda masses [17]. Namely, they are the eigenvalues of a symmetric $r \times r$ matrix M, $M_{kl} = \sum_{j=0}^{r} n_j(\alpha_j)_k(\alpha_j)_l$, or $M = \sum_{j=0}^{r} n_j\alpha_j \otimes \alpha_j$, which encode the integrability of affine Toda field theory. In [17] it is shown for the non-twisted cases that the vector $\mathbf{m} = {}^t(m_1, \ldots, m_r)$, if ordered properly, is the *Perron-Frobenius* eigenvector of the incidence matrix (the Cartan matrix) of the corresponding root system.

The corresponding *quantum* Hamiltonian [1, 16] is

$$H_Q = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \frac{1}{2}\sum_{j=1}^{r}\left[\left(\frac{\partial W(q)}{\partial q_j}\right)^2 + \frac{\partial^2 W(q)}{\partial q_j^2}\right], \tag{4}$$

which is partially solvable or *quasi-exactly solvable* for some affine simple root systems. Namely for $A_{r-1}^{(1)}$, $D_3^{(1)}$, $D_{r+1}^{(2)}$, $C_r^{(1)}$ and $A_{2r}^{(2)}$, the above Hamiltonian (4) is known to have a few exact eigenvalues and corresponding exact eigenfunctions [15].

The polynomials related to the equilibrium position $\bar{q}$ are easy to define for the classical root systems, A, B, C and D. As in the Sutherland cases, we introduce a polynomial having zeros at $\{\sin\bar{q}_j\}$ or $\{\cos 2\bar{q}_j\}$:

$$P_r(q) \propto \prod_{j=1}^{r}(x - \sin\bar{q}_j), \quad \prod_{j=1}^{r}(x - \cos 2\bar{q}_j). \tag{5}$$

For the exceptional root systems, let us choose a set of D vectors $\mathcal{R}$

$$\mathcal{R} = \{\mu^{(1)}, \ldots, \mu^{(D)} \,|\, \mu^{(a)} \in \mathbb{R}^r\},$$

which form a single orbit of the corresponding Weyl group. For example, they are the set of roots Δ itself for simply laced root systems, the set of long (short, middle) roots Δ_L (Δ_S, Δ_M) for non-simply laced root systems and the so-called sets of *minimal weights*. The latter is better specified by the corresponding fundamental representations, which are all the fundamental representations of A_r, the vector ($\mathbf{V}$), spinor ($\mathbf{S}$) and conjugate spinor ($\bar{\mathbf{S}}$) representations of D_r and $\mathbf{27}$ ($\overline{\mathbf{27}}$) of E_6 and $\mathbf{56}$ of E_7. By generalising the above examples (5), we define polynomials

$$P_\Delta^{\mathcal{R}}(x) \propto \prod_{\mu\in\mathcal{R}}\Big(x - \sin(\mu\cdot\bar{q})\Big), \quad \prod_{\mu\in\mathcal{R}}\Big(x - \cos(2\mu\cdot\bar{q})\Big). \tag{6}$$

For more general treatment we refer to our previous article [8].

The resulting polynomials for various affine root systems Π_0 are (we follow the affine Lie algebra notation used in [15, 17]):

$A_{r-1}^{(1)}$ **:** In this case the equilibrium position is exactly the same as that of the A_{r-1} Sutherland [7] and A_{r-1} Ruijsenaars-Sutherland system [12], $\bar{q} = (\pi/2h)\,{}^t(r-1, r-3, \ldots, -(r-1))$ with $h = r$. Thus the polynomial is also the same, the Chebyshev polynomial of the first kind: $2^{r-1}\prod_{j=1}^r\Big(x - \sin\bar{q}_j\Big) = T_r(x) = \cos r\varphi$, $x = \cos\varphi$.

$B_r^{(1)}$ **&** $D_{r+1}^{(2)}$ **&** $A_{2r}^{(2)}$ **:** The Coxeter number is $h = 2r$ for $B_r^{(1)}$, $h = r+1$ for $D_{r+1}^{(2)}$ and $h = 2r+1$ for $A_{2r}^{(2)}$. The equilibrium position is equally spaced $\bar{q} = (\pi/h)\,{}^t(r, r-1, \ldots, 1)$. We obtain the Chebyshev polynomial of the second kind, $U_n(x) = \sin(n+1)\varphi/\sin\varphi$, $x = \cos\varphi$, for $B_r^{(1)}$ and a product of them for $D_{r+1}^{(2)}$ and a sum of them for $A_{2r}^{(2)}$,

$$2^{r-1}\prod_{j=1}^r(x - \cos 2\bar{q}_j) = \begin{cases} (x+1)U_{r-1}(x), & B_r^{(1)}, \\ (x+1)U_{r/2}(x)U_{(r-2)/2}(x) + 1/2, & D_{r+1}^{(2)}, \quad r: \text{ even}, \\ (x+1)U_{(r-1)/2}(x)^2, & D_{r+1}^{(2)}, \quad r: \text{ odd}, \\ \left(U_r(x) + U_{r-1}(x)\right)/2, & A_{2r}^{(2)}. \end{cases} \tag{7}$$

$C_r^{(1)}$ **&** $A_{2r-1}^{(2)}$ **:** The Coxeter number is $h = 2r$ for $C_r^{(1)}$ and $h = 2r - 1$ for $A_{2r-1}^{(2)}$. The equilibrium position is equally spaced $\bar{q} = (\pi/2h)\,{}^t(2r-1, 2r-3, \ldots, 3, 1)$. We obtain the Chebyshev polynomial of the first kind $T_r(x)$ for $C_r^{(1)}$ and a sum of them for $A_{2r-1}^{(2)}$,

$$2^{r-1}\prod_{j=1}^{r}(x - \cos 2\bar{q}_j) = \begin{cases} T_r(x), & C_r^{(1)}, \\ T_r(x) + T_{r-1}(x), & A_{2r-1}^{(2)}. \end{cases} \tag{8}$$

$D_r^{(1)}$ **:** The Coxeter number is $h = 2(r-1)$ and the equilibrium position is equally spaced $\bar{q} = (\pi/h)\,{}^t(r-1, r-2, \ldots, 1, 0)$. We obtain the Chebyshev polynomial of the second kind

$$2^{r-2}\prod_{j=1}^{r}(x - \cos 2\bar{q}_j) = (x^2 - 1)U_{r-2}(x). \tag{9}$$

$E_6^{(1)}$ **:** The Coxeter number is $h = 12$ and the equilibrium position is not equally spaced $\bar{q} = (\pi/h)\,{}^t(4\sqrt{3}, 4, 3, 2, 1, 0)$. We consider the set of minimal weights $\mathbf{27}$ and the set of positive roots Δ_+, which consists of 36 roots. The polynomials are

$$\begin{aligned} 2^{20}\prod_{\mu\in\mathbf{27}}\Big(x - \sin(\mu\cdot\bar{q})\Big) &= (-1+x)\,x^3\,(1+x)\,(-1+2x)^2\,(1+2x)^2\,(-1+2x^2)^2 \\ &\quad\times(-3+4x^2)^3\,(1-16x^2+16x^4)^2, \qquad (10) \\ 2^{27}\prod_{\mu\in\Delta_+}\Big(x - \cos(2\mu\cdot\bar{q})\Big) &= x^6\,(1+x)^3\,(-1+2x)^6\,(1+2x)^7\,(-3+4x^2)^7. \qquad (11) \end{aligned}$$

$E_7^{(1)}$ **:** The Coxeter number is $h = 18$ and the equilibrium position is not equally spaced $\bar{q} = (\pi/2h)\,{}^t(17\sqrt{2}, 10, 8, 6, 4, 2, 0)$. We consider the set of minimal weights $\mathbf{56}$ and the set of positive roots Δ_+, which consists of 63 roots. The $\mathbf{56}$ is even, *ie* if $\mu \in \mathbf{56}$ then $-\mu \in \mathbf{56}$. The positive part of $\mathbf{56}$ is denoted as $\mathbf{56}_+$. The polynomials are

$$\begin{aligned} 2^{24}\prod_{\mu\in\mathbf{56}_+}\Big(x - \cos(2\mu\cdot\bar{q})\Big) &= x^4\,(-3+4x^2)^3\,(-3+36x^2-96x^4+64x^6)^3, \qquad (12) \\ 2^{59}\prod_{\mu\in\Delta_+}\Big(x - \cos(2\mu\cdot\bar{q})\Big) &= (1+x)^4\,(-1+2x)^7\,(1+2x)^7 \\ &\quad\times(-1+6x+8x^3)^8\,(1-6x+8x^3)^7. \qquad (13) \end{aligned}$$

$E_8^{(1)}$ **:** The Coxeter number is $h = 30$ and the equilibrium position is not equally spaced $\bar{q} = (\pi/h)\,{}^t(23, 6, 5, 4, 3, 2, 1, 0)$. We consider the set of positive roots Δ_+, which consists of

120 roots. The polynomial is

$$
\begin{aligned}
2^{116} \prod_{\mu\in\Delta_+} \Big(x - \cos(2\mu\cdot\bar{q})\Big) = \\
&(1+x)^4\,(-1+2x)^8\,(1+2x)^8\,(-1-2x+4x^2)^8\,(-1+2x+4x^2)^8 \\
&\times(1+8x-16x^2-8x^3+16x^4)^8\,(1-8x-16x^2+8x^3+16x^4)^9. \qquad (14)
\end{aligned}
$$

$F_4^{(1)}$ & $E_6^{(2)}$: The Coxeter number is $h=12$ for $F_4^{(1)}$ and $h=9$ for $E_6^{(2)}$ and the equilibrium position is not equally spaced $\bar{q}=(\pi/h)\,{}^t(8,3,2,1)$. We consider the set of long positive roots Δ_{L+} and short positive roots Δ_{S+}, both of which consist of 12 roots reflecting the self-duality of F_4 Dynkin diagram. The polynomials for $F_4^{(1)}$ are

$$2^9 \prod_{\mu\in\Delta_{S+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; x^2(1+x)\,(-1+2x)^2\,(1+2x)^3\,(-3+4x^2)^2, \qquad (15)$$

$$2^9 \prod_{\mu\in\Delta_{L+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; x^2(1+x)\,(-1+2x)^2\,(1+2x)\,(-3+4x^2)^3. \qquad (16)$$

The polynomials associated with the twisted affine root system $E_6^{(2)}$ are

$$2^{12} \prod_{\mu\in\Delta_{S+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; (1+2x)^3\,(1-6x+8x^3)^3, \qquad (17)$$

$$2^{12} \prod_{\mu\in\Delta_{L+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; 2\,(-1+x)\,(1+2x)^2\,(1-6x+8x^3)^3. \qquad (18)$$

$G_2^{(1)}$ & $D_4^{(3)}$: The Coxeter number is $h=6$ for $G_2^{(1)}$ and $h=4$ for $D_4^{(3)}$ and the equilibrium position is $\bar{q}=(\pi/2h)\,{}^t(3\sqrt{6},\sqrt{2})$. We consider the set of long positive roots Δ_{L+} and short positive roots Δ_{S+}, both of which consists of 3 roots reflecting the self-duality of G_2 Dynkin diagram. The polynomials for the untwisted $G_2^{(1)}$ are

$$2^3 \prod_{\mu\in\Delta_{S+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; 2\,(1+x)\,(-1+2x)\,(1+2x), \qquad (19)$$

$$2^3 \prod_{\mu\in\Delta_{L+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; (-1+2x)^2\,(1+2x). \qquad (20)$$

The polynomials for the twisted $D_4^{(3)}$ are

$$\prod_{\mu\in\Delta_{S+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; x^2\,(1+x), \qquad (21)$$

$$\prod_{\mu\in\Delta_{L+}} \Big(x-\cos(2\mu\cdot\bar{q})\Big) \;=\; x^2\,(-1+x). \qquad (22)$$

Before closing this paper, let us briefly remark on the identities arising from *foldings* of root systems. Among them those relating two un-twisted root systems, *ie* with superscript (1) are quite simple.

Folding $A^{(1)}_{2r-1} \to C^{(1)}_r$ **:** The vector weights of A_{2r-1} ($2r$ dim.) become those of C_r ($2r$ dim.). This relates T_{2r} to T_r in (8) as

$$A_{2r-1}: \quad T_{2r}(x) = (-1)^r \, T_r(1-2x^2), \quad C^{(1)}_r. \tag{23}$$

Folding $D^{(1)}_{r+1} \to B^{(1)}_r$ **:** This gives a quite obvious relation as seen from (9) and (7).

Folding $E^{(1)}_6 \to F^{(1)}_4$ **:** In this folding the minimal weights **27** of E_6 become Δ_S (24 dim.) of F_4 plus three zero weights. Thus we obtain

$$E^{(1)}_6: \quad 2\,(10)/x^3 = (15)_{x\to 1-2x^2}, \quad F^{(1)}_4. \tag{24}$$

We also obtain

$$E^{(1)}_6: \qquad (11) = (15)^2 \times (16), \qquad F^{(1)}_4, \tag{25}$$

since the 72 roots of E_6 are decomposed into $2\Delta_S + \Delta_L$ (24 dim.) of F_4.

Folding $D^{(1)}_4 \to G^{(1)}_2$ **:** The vector weights of D_4 (8 dim.) decompose into Δ_S (6 dim.) plus two zero weights of G_2 leading to the identity

$$D^{(1)}_4: \qquad 2\,(9)_{r=4}/(x-1) = (19), \quad G^{(1)}_2. \tag{26}$$

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